

On Quantum Cohomology

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Abstract We discuss a general quantum theoretical example of quantum cohomology and show that various mathematical aspects of quantum cohomology have quantum mechanical and also observable significance.

The quantum cohomology is one of the most fundamental and interesting mathematical-physical fields and although it is introduced according to certain physical models [1], however it should be considered as a general invariant geometrical tool for all quantum theories. Nevertheless, in view of various mathematical difficulties [2] its physical foundations are not well discussed yet. The main reason for this situation lies on the non-well understood topological or differential geometric structure of quantization as a general scheme.

It is important to mention that if one takes the fact seriously that classical mechanics is a classical limit of quantum mechanics, then a fundamental part of topology which is based on the globalization of classical mechanical results, e. g. Morse theory and symplectic topology, should be considered as a classical limit of some quantum topological originals. In view of the fact that the main difference between quantum and classical mechanics is the global (topological) character of states and accordingly the observables of quantum mechanics despite of local character of classical observables [3]; It is natural that the main difference in the classical and quantum geometries also arises in the topological scope. In other words, in view of the genuine topological character of quantization it is quite natural that quantization has such an influence like a quantum deformation of cohomology on the topology of the quantized system.

Moreover, in view of the necessary symplectic background of quantization it is also not surprising that the quantum cohomology becomes equivalent to some generalization of certain results on invariant structures of symplectic mechanics (in quantum theoretical sense), i. e. to the so called Floer cohomology [4].

Briefly speaking the quantum cohomology should be considered as a result of existence of flat connections, which are related with quantization, together with the multiply connectedness of the quantum phase space which is related with multivalued functions. Equivalently, a closed "path" (circle) surrounding the minimum cell of the quantized phase space with an area $h > 0$ can not be shrunk to a point. It can be considered also as a result of finiteness of some relevant measures like position, i. e. position uncertainty δq which are prevented to become zero in quantum mechanics ($\delta q > 0$). In this sense, for example, the usual notion of "path" or *boundary* loses its definition in quantum phase space and we have a deformation of the original classical homologies and cohomologies, whereas the usual "classical" homologies and cohomologies should be considered as valid only in the classical phase space which is a simply connected manifold. To be precise, let us mention that in view of the uncertainty relation

$\delta q \delta p = \hbar > 0$ a ring of width δq in quantum phase space can not be shrunk to a circle, i. e. to the classical boundary of a $(2-D)$ manifold, whereas this is possible in a classical phase space where $\hbar = 0$. Considering the finiteness of the most minimal energy in quantum mechanics of a harmonic oscillator ($\delta E = 2E_0 > 0$) one should also recall the structure of original Morse function as the main ingredient of invariant structures on manifolds which is an energy functional. Hence, it is the ground state energy which differs the classical cohomology of harmonics, $\Delta Harm^0 = 0$, from the quantum cohomology of ground state, i. e. the Floer cohomology.

Coming back to the question of multiply connectedness it seems also natural to use the so called Novikov ring [2] of multivalued functions to construct the quantum cohomology: Because on the one hand we have the natural relation between the multiply connectedness of a manifold and the multivalued functions and on the other hand we know about the quantization of angular momentum which results from a transformation of the related multivalued function into a single valued one [5]. Thus, the general role of multivalued functions in quantum cohomology should be understood if one recalls that they represent the wave functions before their quantization.

To make further relations of quantum cohomology with physical observable effects transparent let us mention that our interest on this field arose from the question of edge currents [6] and potential drops [7] in quantum Hall effects (QHE) and also from the equivalence between the quantization of Hall measures, i. e. resistivity and conductivity, and the flux quantization in superconductivity [8]. Since, the question of boundary which is the main ingredient of homology plays an essential role in these phenomena, i. e. in the edge current, potential drops and flux quantization. Furthermore, both QHE and the cohomology ring of a $2-D$ manifolds are topologically invariant structures, thus it is relevant to look for some relation between them. However, the first one is a quantum structure, whereas the second one is a classical structure, thus one should ask if there is a quantum version of cohomology which should be related to the topological QHE structure? Therefore, the questions of boundary and also the relative cohomology can be helpful here. Specially, the absence of the notion of "path" in quantum theory which can be considered in the closed case as the boundary of a $(2-D)$ manifold, gives the right instrument to define non-trivial cohomologies.

We will discuss here the general quantum mechanical foundations of quantum cohomology and show

its equivalence with the Floer cohomology. In a subsequent paper [9] we discuss also the mathematical questions which arise in proving this equivalence for the original models [?] and we will clarify the background structure of quantum cohomology of these models according to the discussion in Ref. [2]. Although, the structure of mentioned models of quantum cohomology is (in view of their various "extra" degrees of freedom) rather complicated to show the sufficiency of the pure quantization scheme of these models for the emergence of quantum cohomology directly in these cases. However, we give at least some intuitive reasons for our statement on the fundamentality and generality of the appearance of quantum cohomology in all *quantized* theories which can be proved.

First, recall that at least in the $(2 + 1)$ dimensional case the supersymmetry should be considered as a result of Poincare duality of $H^2 \cong H^0$ of the quantized theory. In other words, a supersymmetry requirement in suitable degrees of freedom manifests such a Poincare duality which is essential for a consistent quantum structure and also for a quantum cohomology. Therefore, the reason why such supersymmetric models [?] demonstrate quantum cohomologies should lie in their genuine (constructed) Poincare duality which is related with their geometric quantization.

Secondly, in view of the fact that every quantization scheme has to be equivalent to the canonical quantization, we have a classical symplectic background for all theories which are *quantized*. On the one hand, we know from the geometric quantization [10] that quantization of a classical phase space can be given according to the structure of the (complex) line bundle over the phase space which is equivalent to a principal $U(1)$ -bundle over the same. On the other hand, a quantization of a symplectic structure of the classical phase space is equivalent to its complexification at least in the Heisenberg approach which is equivalent to the establishing of a Kaehler structure on the classical phase space. Thus, we have by the quantization of a theory a transition from the classical cohomology as the invariant structure of its classical phase space to an other invariant structure of its quantized phase space. It is this invariant structure which should be considered as the quantum cohomology for the given theory, hence its classical limit becomes the mentioned classical cohomology. In other words, the quantum cohomology is a topologically invariant result of the quantum deformation on a given phase space.

To show these circumstances in mentioned models, let us mention that for example in the first model the Kaehler structure and a $H^1 = 0$ requirement are used to obtain the "invariants" of related quantum

cohomology. Now $H^1 = 0$ means that one is considering *only* flat connections or properly the moduli space of instantons of the model. Flat connections in turn means (in the proper $U(1)$ sense) that we have to do with the quantization scheme according to the flat $U(1)$ -connection.

Moreover, in this model the basic "quantum cohomological" invariants are considered as reducible to integrals over the moduli space of instantons which is nothing more than the *quantizable* phase space of model. Thus, we are more or less concerning the quantum structure of the phase space of the model. Hence, the symplectic structure of the same moduli space/phase space is that structure which (after its quantization) allows one to define such quantum cohomological invariants.

In the second model in Ref. [?] the main structures used to identify the quantum cohomology (ring) of the theory are the existent Kaehler/Calabi-Yau structure and the $U(1)$ current of model. Recall that the calabi-Yau structure can be considered (in the quantization scheme) as equivalent to the (holomorphic) polarization of phase space where the symplectic form vanishes. It should be mentioned that the holomorphic polarization can be considered again as the flatness condition of the $U(1)$ -connection of the quantization.

Thus, depending on the type of quantization we have various apparently independent conditions or geometrical structures on the phase space which are synonyms of each others in a more fundamental theory of quantization.

Thus, properly here also it is only the abstract quantum structure of the quantized phase space of the theory which determine the existence and the general abstract structure of the related quantum cohomology. Therefore, every quantized theory should have its own type of quantum cohomology, however the existence and abstract structure of these quantum cohomologies should depend only on the general quantization structure, i.e. the line bundle/ $U(1)$ or the Kaehler/Clabi-Yau structure of the related phase spaces.

To begin let us mention some usefull results on the classical cohomology that according to the celebrated de Rham's theorem one has $H^r \cong H_r$ and by the Hodge's theorem on a compact orientable Riemannian manifold we have $H^r(M) \cong Harm^r(M)$, where the $Harm^r$ is defined by $\Delta Harm^r = 0$. Moreover, we have according to the Poincare duality on a compact m -dimensional maifold $H^m \cong H^0$. On the other hand, we know from the most simple non-relativistic quantum mechanics that for the ground state

$\Delta|0\rangle = E_0|0\rangle$. Thus, in view of the fact that always $E_0 \propto \hbar$ and that the classical limit of quantum mechanics is related with $\hbar \rightarrow 0$ [?, ?], one has $\{|0\rangle_{\hbar \rightarrow 0}\} \in Harm^0$.

Therefore, one has for example for H^0 cohomology the isomorphism $H^0 \cong Harm^0 \cong H_{Q,\hbar \rightarrow 0}^0$, where $H_Q^0 = H_{\psi_0}$ is the cohomology of ground state. In this sense one should obtain the usual "classical" cohomology as the classical limit of quantum cohomology, i. e. $H^m \cong Harm^m \cong H_{Q,\hbar \rightarrow 0}^m$ or one may consider the ground state as a quantum deformation of harmonic functions. Nevertheless, one can also consider that the quantum Laplace operator is a deformation of the usual Laplacian $\Delta_Q := \Delta + O(\hbar)$, whereas the ground state remains a harmonic function. Thus, one has at any case according to the ground state equation $\Delta|0\rangle = E_0|0\rangle$ a deformed cohomology ring structure in the quantum case by the Hodge's theorem.

Now let us consider for the spatial base manifold of our quantum theory a Riemann surface Σ with boundary, e. g. a disc. This is for example the case if we use to quantize a classical Chern-Simons-theory for QHE which is defined on $\Sigma \times R$ [8]. Despite of classical case where the "classical" boundary of $\partial\Sigma = C_1$ is well defined and we should have a classical mechanical prescription to determine such an "absolute" boundary. In quantized cases, i. e. if we have a quantum theory on Σ , the notion of boundary of Σ_Q loses its definition and we have no quantum theoretical prescription to define or to measure the boundary $\partial\Sigma_Q$. In other words, if Σ_Q is the polarized/quantized phase space, then in view of the uncertainty relation $\delta p \cdot \delta q = \hbar$ we have always $\delta q > 0$ and so it is $\partial\Sigma_Q \neq C_1$ [?]. Recall that the notion of polarization becomes equivalent to that of holomorphicity in the suitable almost complex or Kaehler cases, where the notion of J-holomorphic curves enters the quantum cohomology.

Therefore, the Σ_Q is in quantized cases boundaryless in the usual "classical" sense of boundary or it is $\partial\Sigma_Q = \emptyset$. We are faced with such situation in the QHE, where the edge currents which should be classically exactly on the boundary of samples, are defined to flow within a width of magnetic length l_B on the boundary of sample. Furthermore, also in QHE there is a potential drop around the boundary in a width of l_B^{-1} which should not exist in classical case [11]. These effects can be understood if one takes the electrodynamical uncertainty into account [?] where we have $e\delta A_m \cdot \delta q = \hbar$ with $\delta q := l_B$ which is confirmed not only by the flux quantization but also by definition of the magnetic length itself. Thus, in QED cases, e. g. in QHE which is the best known quantum effect in two dimensions, either one

has to determine a new prescription to define the boundary of Σ_Q or one may use the known methodes to define it with the help of edge currents or potential drops. At any case as it is mentioned above in quantum cases there is no possibility to define an absolute *one dimensional* boundary like the classical boundary C_1 for Σ_Q . The quantum measurements and all possible quantum prescriptions can determine only a "quantum" boundary for Σ_Q which is a two dimensional ring of width $\delta q = l_B$.

It should be mentioned that, of course one is able to define a classical boundary for Σ with the help of a classical theory, however this prescription and such a boundary are exact only within the classical limit (also of a quantum measurement) and the whole system is purely *classical*.

Now we use the above mentioned circumstances to show first the existence of a non-trivial cohomology in quantum case which becomes trivial in the classical case and second to prove its isomorphism with a related Floer cohomology. By the Floer cohomology we mean here the general cohomology of the ground state of the quantum system under consideration. If it is so, then in view of the already proved isomorphism between the Floer and quantum cohomology [2] one should consider the mentioned non-trivial cohomology as a candidate of quantum cohomology.

Roughly speaking the quantum cohomology H_Q^r is given by $H_Q^r := H^r + (\text{additional terms})$, nevertheless in view of the Hodge's theorem and the definition of $Harm^r$ one can use alternatively the deformation of the Laplace operator to define the quantum cohomology. We show that there is a non-trivial maximal cohomology H_Q^2 on Σ_Q which is isomorphic to a H_Q^0 or to $Harm^0$, which is a $U(1)$ -Floer cohomology.

Therefore, let us first define the above mentioned general Floer cohomology H_F according to the cohomology of ground state, i. e.

$$H_F := \{|0\rangle, \quad \Delta|0\rangle = E_0|0\rangle\} \quad ; \quad E_0 \propto \hbar \quad (1)$$

It is obviously a deformation of the cohomology of harmonic functions which is isomorphic to H^0 and in our case also it is also isomorphic to the H^2 . Furthermore, it results in a deformation of the Laplace operator

$$\Delta_Q := \Delta + O(\hbar) = \Delta - E_0, \quad (2)$$

in the sense that now we have

$$\Delta_Q |0\rangle = 0 \quad (3)$$

It means that the deformed Laplacian has again the harmonic functions as its eigen vectors

$\Delta_Q Harm^0 = 0$ or $|0\rangle \in Harm^0$. Furthermore, it requires also a deformations of the exterior differential operator and its adjoint:

$$d_Q := d + O(\hbar^{\frac{1}{2}}), \quad d_Q^\dagger := d^\dagger + O(\hbar^{\frac{1}{2}}) \quad (4)$$

This deformation requires also a deformation of the cup product which is essential in the usual definition of quantum cohomology and it can result also in a deformed differential structure of the quantum plane- or quantum group types [9].

More important is the fact that, in view of the above analysis of quantum situation with $\partial\Sigma_Q = \emptyset$ we have a non-trivial maximal cohomology on Σ_Q which is given according to the trivially closed but non-exact electromagnetic 2-form:

$$H_Q^2(\Sigma_Q; F) = \{F; dF = 0; F \neq dA\} := \{dF = 0 / F = dA\} \quad (5)$$

Obviously, such a cohomology can be defined also for any relevant general two form, i. e. for Ω^2 , on Σ_Q instead of F .

Recall that this cohomology is trivial in the classical case where $\delta q = 0$ and we have $\partial\Sigma = C_1$. In this case as it is well known the closed 2-form F can always be considered as $F = dA$ or $\Omega^2 = d\Omega^1$. Thus, we have a classically trivial cohomology which become only in quantum case non-trivial. Recall further that according to the Hodge's decomposition in its 2-D case $\Omega^2 = d\Omega^1 \oplus Harm^2$ which applies also for our case, the closed form F should be written as $F = dA \oplus Harm^2$.

Moreover, a deformation like (2) recalls one on the Witten's supersymmetric modification of Laplacian [12] which should be discussed later.

Of course to show that H_Q^2 is the same as the known quantum cohomology [?] one should prove for example that the space of 2-forms $\{F\}$ on the $U(1)$ bundle is the same as the space of J-holomorphic maps between Riemann surface and the base manifold of the discussed quantum theory [13]. This will be in our terminology a map between the Riemann surface and the polarized phase space of the quantum theory. We will prove this in a subsequent paper [9], however let us mention that in our case of $U(1)$ bundle we have a map from Riemann surface to the 2-D phase space or the moduli space of the $U(1)$ -connections which is again a Riemann surface in view of its Kaehler structure.

Moreover, as it is mentioned we have in view of the Hodge's theorem

$$H_Q^2 \cong Harm^2 \quad (6)$$

and according to the Poincare duality

$$H_Q^2 \cong H_Q^0 \quad (7)$$

Thus, we obtain the desired isomorphism between our quantum mechanically non-trivial cohomology H_Q^2 and the Floer cohomology of ground state $|0\rangle \in Harm^0$ by the use of Hodge's theorem for $H^0 \cong Harm^0$:

$$H_Q^2 \cong Harm^0 \cong H_F \quad (8)$$

As a conclusion we like to mention that the quantum theoretical version of the above mathematical prove of these equivalencies should be demonstrated as follows:

- *) H_Q^2 is a result of $\delta q > 0$ in quantum theory.
- *) $\Delta_Q Harm^0 = E_0 Harm^0$ or $\Delta_Q |0\rangle = E_0 |0\rangle$ is a result of $\delta E = 2E_0$ in quantum theory.
- *) $H_Q^2 \cong Harm^0 \cong H_F$ isomorphism is a result of the uncertainty relation: $\delta q \cdot \delta p \cong \delta E \cdot \delta t = \hbar$.

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[13] I am thankful to Prof. Y. I. Manin to mention this point to me concerning the space of connections on the $U(1)$ bundle. I mention here a modified version of his remark concerning the space of $2 - Forms$ which seems to me in good relation with the second homology class involved in the J-holomorphic curves.

Futhermore, I am thankful for his *provisional* agreement with the idea that in general quantum cohomological structures should be possible in every *quantized* theory (private communications).